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Output-feedback control of a class of stochastic nonlinear systems with linearly bounded unmeasurable states

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SUMMARY

In this paper, the problems of stochastic disturbance attenuation and asymptotic stabilization *via* output feedback are investigated for a class of stochastic nonlinear systems with linearly bounded unmeasurable states. For the first problem, under the condition that the stochastic inverse dynamics are generalized stochastic input-to-state stable, a linear output-feedback controller is explicitly constructed to make the closed-loop system noise-to-state stable. For the second problem, under the conditions that the stochastic inverse dynamics are stochastic input-to-state stable and the intensity of noise is known to be a unit matrix, a linear output-feedback controller is explicitly constructed to make the closed-loop system globally asymptotically stable in probability. Using a feedback domination design method, we construct these two controllers in a unified way. Copyright © 2007 John Wiley & Sons, Ltd.

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KEY WORDS: stochastic nonlinear systems; stochastic input-to-state stability; output-feedback stabilization; backstepping design method; noise-to-state stability

1. INTRODUCTION

The design of global stabilization controller for stochastic nonlinear systems has been an active area of research in recent years ([1–3] and the references therein). Since Deng and Krstić [4] firstly gave a result of output-feedback stabilization, the output-feedback controller design for stochastic nonlinear systems has received more intensive investigation [2, 5–9], which is because not only in general, the design of output-feedback control is more difficult and challenging than that of full state-feedback control, but also the output-feedback control is more practical in

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engineering. However, these known results are limited to the systems in output-feedback form, in which the nonlinear terms only depend on the measured output. For the deterministic systems, in [10] counterexamples were given indicating global stabilization of the nonlinear systems in general low-triangular form *via* output feedback is usually impossible without introducing extra growth conditions on the unmeasurable states of the system. Since then, much research work has been focused on the output-feedback global stabilization of nonlinear systems under various structure or growth conditions [11–16]. Recently, there are some results of output-feedback control for the stochastic nonlinear systems in which nonlinear terms are dependent on the output and unmeasurable inverse dynamics or unmodeled dynamics [3, 17, 18]. But for stochastic nonlinear systems dependent on general unmeasurable states, to the authors' knowledge, there is no related result.

In this paper, we consider disturbance attenuation and asymptotic stabilization *via* output feedback for a class of stochastic nonlinear systems in which drift and diffusion terms depend on unmeasurable states besides the output and unmeasurable inverse dynamics. Firstly, to deal with stochastic inverse dynamics, two stability concepts are introduced: stochastic input-to-state stable (SISS) with respect to the stochastic input and generalized SISS (GSISS) with respect to the stochastic input and unknown covariance of noise. Under the assumption that the inverse dynamics are GSISS, a linear output-feedback controller is explicitly constructed to make the closed-loop system noise-to-state stable; when the intensity of noise is known to be unit matrix, under the assumption that the inverse dynamics are SISS, a linear output-feedback controller is explicitly constructed to make the closed-loop system globally asymptotically stable in probability. Based on a feedback domination design method, a unified design procedure for the above two controllers is supplied.

The remainder of the paper is organized as follows. Section 2 provides some notations and preliminaries. Section 3 describes the problem to be investigated. Section 4 presents the design of high gain observer. The output-feedback control design procedure is given in Section 5. Stability analysis of the closed-loop system in question is given in Section 6. Section 7 contains some concluding remarks.

2. NOTATIONS AND PRELIMINARIES

The following notations will be used throughout this paper. $\mathbb{R}_+$ denotes the set of all nonnegative real numbers; $\mathbb{R}^n$ denotes the real n-dimensional space; $\mathbb{R}^{n\times r}$ denotes the real $n\times r$ matrix space. For a given vector or matrix X, X^{T} denotes its transpose; $\mathrm{Tr}(X)$ denotes its trace when X is square; $|X|$ denotes the Euclidean norm of a vector X; $||X||$ denotes the Frobenius norm of matrix X defined by $||X|| = \sqrt{\mathrm{Tr}(X^{\mathrm{T}}X)}$; $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote the minimal and maximal eigenvalue of symmetric real matrix X, respectively; $\mathscr{C}^i$ denotes the set of all functions with continuous ith partial derivatives; $\mathscr{C}^{2,1}(\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+)$ denotes the family of all nonnegative functions $V(x,t)$ on $\mathbb{R}^n\times\mathbb{R}_+$ which are $\mathscr{C}^2$ in x and $\mathscr{C}^1$ in t; $\mathscr{K}$ denotes the set of all functions: $\mathbb{R}_+\to\mathbb{R}_+$, which are continuous, strictly increasing and vanish at zero; $\mathscr{K}_\infty$ denotes the set of all functions which are of class $\mathscr{K}$ and unbounded; $\mathscr{K}\mathscr{L}$ denotes the set of all functions $\beta(s,t)$: $\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$, which is of $\mathscr{K}$ for each fixed t, and decreases to zero as $t\to\infty$ for each fixed s.

For a given stochastic system

$$\mathrm{d}x = (f(x,t) + g(x,t)u)\,\mathrm{d}t + h(x,t)\,\mathrm{d}w$$

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define a differential operator $\mathscr{L}$ as follows:

$$\mathscr{L}V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) + \frac{\partial V}{\partial x} g(x,t)\,u + \frac{1}{2}\mathrm{Tr}\left\{ h^{\mathrm{T}}(x,t)\frac{\partial^2 V}{\partial x^2} h(x,t)\right\}$$

where $V(x,t) \in \mathscr{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$; $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input; $f \in \mathbb{R}^n$, $g \in \mathbb{R}^n$ and $h \in \mathbb{R}^{n\times r}$ are locally Lipschitz functions; w is an r-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, P)$ with Ω being a sample space, $\mathscr{F}$ being a σ-field, and P being the probability measure.

For control-free stochastic nonlinear systems of the form:

$$\mathrm{d}x = f(x,t)\,\mathrm{d}t + h(x,t)\,\mathrm{d}w \tag{1}$$

the following stability notions introduced in [19] will be used in the paper.

Definition 1
For system (1) with $f(0,t) \equiv 0$ and $h(0,t) \equiv 0$, the solution $x(t) \equiv 0$ is said to be globally asymptotically stable in probability (GASiP), if for any given $\epsilon \in (0,1)$, there exists a function $\beta(\cdot,\cdot) \in \mathscr{K}\mathscr{L}$ such that

$$P\{|x(t)| < \beta(|x_0|, t)\} \geqslant 1-\epsilon, \quad \forall t \geqslant 0, \;\; \forall x(0) = x_0 \in \mathbb{R}^n \backslash \{0\}$$

The following theorem gives sufficient conditions on the stability introduced above.

Theorem 1 (Krstić and Deng [19])
For system (1), assume that $f(x,t), h(x,t)$ are locally Lipschitz in x uniformly in t. If there exists a function $V(x,t) \in \mathscr{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, which is positive definite and radially unbounded in x uniformly in t; a constant $c \geqslant 0$, and a positive definite function $W(x)$ such that

$$\mathscr{L}V \leqslant -W(x) + c$$

then

(a) there exists an almost surely unique solution on $[0,\infty)$;
(b) the zero solution of system (1) is GASiP, when $f(0,t) \equiv 0, h(0,t) \equiv 0$ and $c = 0$.

Consider the following stochastic nonlinear systems:

$$\mathrm{d}x = f(x,v,t)\,\mathrm{d}t + g(x,v,t)\Sigma(t)\,\mathrm{d}w \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $v = v(x,t): \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^m$ is the input, $\Sigma: \mathbb{R}_+ \to \mathbb{R}^{r\times r}$ is a Borel bounded measurable function and the matrix $\Sigma(t)$ is nonnegative definite for each $t \geqslant 0$, w is an r-dimensional standard Brownian motion defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, P)$ with $\{\mathscr{F}_t\}_{t\geqslant 0}$ being a filtration; $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}^{n\times r}$ are assumed to be locally Lipschitz in (x,v) uniformly in t. Assume that for every initial condition $x(0) = x_0$, each essentially bounded measurable input v and Borel bounded measurable function $\Sigma(t)$, system (2) has a unique solution $x(t)$ on $[0,\infty)$ which is $\mathscr{F}_t$-adapted, t-continuous, and measurable with respect to $\mathscr{B} \times \mathscr{F}$, where $\mathscr{B}$ denotes the Borel σ-algebra of $\mathbb{R}$ [20]. Then, we have the following definitions.

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Definition 2
System (2) with $\Sigma(t)=I$ is SISS if for any given $\epsilon\in(0,1)$, there exists a $\mathscr{KL}$ function $\beta(\cdot,\cdot)$ and a $\mathscr{K}$ function $\gamma(\cdot)$ such that

$$P\left\{|x(t)|<\beta(|x_0|,t)+\gamma\left(\sup_{0\leqslant s\leqslant t}||v_s||\right)\right\}\geqslant 1-\epsilon,\quad \forall t\geqslant 0,\ \ \forall x_0\in\mathbb{R}^n\backslash\{0\} \tag{3}$$

where $||v_s||=\inf_{\mathscr{A}\subset\Omega,P(\mathscr{A})=0}\sup\{|v(x(\omega,s),s)|:\omega\in\Omega\backslash\mathscr{A}\}$.

Definition 3
System (2) is GSISS if for any given $\epsilon\in(0,1)$, there exists a $\mathscr{KL}$ function $\beta(\cdot,\cdot)$ and $\mathscr{K}$ functions $\gamma(\cdot)$ and $\gamma_w(\cdot)$ such that

$$P\left\{|x(t)|<\beta(|x_0|,t)+\gamma\left(\sup_{0\leqslant s\leqslant t}||v_s||\right)+\gamma_w\left(\sup_{0\leqslant s\leqslant t}||\Sigma(s)\Sigma(s)^{\mathrm{T}}||\right)\right\}\geqslant 1-\epsilon,\quad \forall t\geqslant 0,\ \ \forall x_0\in\mathbb{R}^n\backslash\{0\}$$

Remark 1
Different from all the existing concepts characterizing the SISS behavior, here the input v in system (2) is assumed to be a function of t and x, precisely, $v=v(x(\omega,t),t)$, and can be regarded as a Markov control input, which ensures the corresponding solution process $x(\omega,t)$ is an Itô diffusion, and hence, a Markov process [21]. This kind of input is the most general one for the systems described by Itô diffusion stochastic differential equations, for which global control design has been a hot topic of research in recent years (see [2, 22] and the references therein). The above definitions are generalization of NSS [19]; and when $v(x,t)=v(t)$ is deterministic and $\Sigma(t)=I$, Definition 2 is the SISS given in [23].

Remark 2
In system (2), $\Sigma(t)$ indicates the intensity of the system noise. In practical systems, $\Sigma(t)$ exists widely. For instance, in the model of stock price

$$\mathrm{d}p(t)=p(t)[b(t)\,\mathrm{d}t+\Sigma(t)\,\mathrm{d}w(t)]$$

$p(t)=[p_1(t),\ldots,p_n(t)]^{\mathrm{T}}$ where $p_i(t)$ is the price per share of the ith stock at time t, and $\Sigma(t)=(\sigma_{ij}(t))$ where σ_{ij} is called 'volatility coefficient' and models the instantaneous intensity with which the jth source of uncertainty influences the price of the ith stock at time t (see e.g. [24, 25]).

The following theorem and corollary provide sufficient conditions on GSISS and SISS, respectively.

Theorem 2
For system (2), assume that $f(x,v,t), g(x,v,t)$ are locally Lipschitz in (x,v) uniformly in t. If there exists a function $V(x,t)\in\mathscr{C}^{2,1}(\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+)$ and $\mathscr{K}_\infty$ functions $\alpha_1,\alpha_2,\alpha,\chi,\chi_w$ such that

$$\alpha_1(|x|)\leqslant V(x,t)\leqslant\alpha_2(|x|) \tag{4}$$

$$\mathscr{L}V\leqslant-\alpha(|x|)+\chi(|v|)+\chi_w(||\Sigma\Sigma^{\mathrm{T}}||) \tag{5}$$

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then,

(a) system (2) is GSISS;
(b) if $v=0$, system (2) is NSS.

Proof
See Appendix A. □

Similar to the proof of Theorem 2, we can obtain the following result.

Corollary 1
For system (2) with $\Sigma(t)\equiv I$, assume that $f(x,v,t), g(x,v,t)$ are locally Lipschitz in (x,v) uniformly in t. If there exists a function $V(x,t)\in\mathscr{C}^{2,1}(\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+)$ and $\mathscr{K}_\infty$ functions α_1,α_2, α,χ such that

$$\alpha_1(|x|)\leqslant V(x,t)\leqslant\alpha_2(|x|)$$
$$\mathscr{L}V\leqslant -\alpha(|x|)+\chi(|v|)$$

then system (2) is SISS.

Remark 3
From Theorem 2 and Corollary 1, it can be seen that the sufficient conditions of (generalized) SISS are similar to their deterministic counterparts [11, 14, 26]. But the (generalized) SISS is applicable for more general inputs, including the general deterministic input $v(t)$, the intensity $\Sigma(t)$ of the noise, or even the stochastic process input $v(x,t)$. In addition, its analysis is more complex and difficult, which can be seen from the proof of Theorem 2 and the following problems to be investigated.

3. PROBLEM FORMULATION

Consider the following stochastic system:

$$\mathrm{d}x_z = f_0(x_z,y,t)\,\mathrm{d}t + g_0(x_z,y,t)\Sigma(t)\,\mathrm{d}w \tag{6}$$

$$\begin{aligned}
\mathrm{d}x_1 &= (x_2+f_1(x_1,x_z,t))\,\mathrm{d}t + g_1(x_1,x_z,t)\Sigma(t)\,\mathrm{d}w\\
&\ \ \vdots\\
\mathrm{d}x_{n-1} &= (x_n+f_{n-1}(\bar{x}_{n-1},x_z,t)\,\mathrm{d}t + g_{n-1}(\bar{x}_{n-1},x_z,t)\Sigma(t)\,\mathrm{d}w\\
\mathrm{d}x_n &= (u+f_n(\bar{x}_n,x_z,t)\,\mathrm{d}t + g_n(\bar{x}_n,x_z,t)\Sigma(t)\,\mathrm{d}w\\
y &= x_1
\end{aligned} \tag{7}$$

where $x=[x_1,\dots,x_n]^{\mathrm{T}}$ is the state, $u\in\mathbb{R}$ is the control input, $y\in\mathbb{R}$ is the measured output; $x_z\in\mathbb{R}^m$ is the state of the unmeasurable inverse dynamics; $\bar{x}_i=[x_1,\dots,x_i]^{\mathrm{T}}$, f_i,g_i, $i=0,\dots,n$, are locally Lipschitz in the first two arguments uniformly in t and satisfy

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$f_i(0,0,t) \equiv 0, g_i(0,0,t) \equiv 0$; $\Sigma : \mathbb{R}_+ \to \mathbb{R}^{r\times r}$ is a Borel bounded measurable function and $\Sigma(t)$ is nonnegative definite for each $t \geqslant 0$.

Consider the following two groups of assumptions. The first group is for the disturbance attenuation problem, while the second one is for the asymptotic stabilization problem.

A1: There exist known positive constants C_f, C_g, and K_f such that for any $i = 1, \ldots, n$,

$$|f_i(\bar{x}_i, x_z, t)| \leqslant C_f(|x_1| + \cdots + |x_i|) + K_f|x_z|$$
$$|g_i(\bar{x}_i, x_z, t)| \leqslant C_g$$

A2: There exist known positive constants α, γ, γ_w and a function $V_z(x_z, t) \in \mathscr{C}^{2,1}(\mathbb{R}^m \times \mathbb{R}_+; \mathbb{R}_+)$, which is positive definite and radially unbounded in x_z uniformly in t, such that

$$\mathscr{L}V_z \leqslant -\alpha|x_z|^2 + \gamma|y|^2 + \gamma_w||\Sigma||^2 \tag{8}$$

B1: $\Sigma(t) \equiv I$.

B2: There exist known positive constants C_f, C_g, K_f, and K_g such that for any $i = 1, \ldots, n$,

$$|f_i(\bar{x}_i, x_z, t)| \leqslant C_f(|x_1| + \cdots + |x_i|) + K_f|x_z|$$
$$|g_i(\bar{x}_i, x_z, t)| \leqslant C_g(|x_1| + \cdots + |x_i|) + K_g|x_z|$$

B3: There exist known positive constants α, γ and a function $V_z(x_z, t) \in \mathscr{C}^{2,1}(\mathbb{R}^m \times \mathbb{R}_+; \mathbb{R}_+)$, which is positive definite and radially unbounded in x_z uniformly in t, such that

$$\mathscr{L}V_z \leqslant -\alpha|x_z|^2 + \gamma|y|^2 \tag{9}$$

In this paper, the following two problems are to be solved.

3.1. Disturbance attenuation problem

For system (6)–(7), under Assumptions A1 and A2, the control objective is to design a smooth dynamic output-feedback controller

$$\dot{\chi} = \varpi(\chi, y)$$
$$u = \mu(\chi, y) \tag{10}$$

such that the closed-loop system consisting of (6), (7) and (10) is stochastic disturbance attenuation in the NSS sense [1].

3.2. Asymptotic stabilization problem

For system (6)–(7), under Assumptions B1–B3, the control objective is to design a smooth dynamic output-feedback controller (10) such that the closed-loop system consisting of (6), (7) and (10) is GASiP.

Remark 4

(i) Due to the existence of unmeasurable states $x_2, \ldots, x_i$ in nonlinear terms f_i and unknown covariance of the noise, the boundedness of the diffusion terms g_i is assumed in

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Assumption A1. This technical assumption, similar to that in [7], is required to bound the estimate error by appropriate terms, which will be clear later.

(ii) By Theorem 2 and Assumption A2, the inverse dynamic (6) is GSISS. By Corollary 1 and Assumption B3, the inverse dynamic (6) is SISS with respect to the virtual input y. For the unmeasurable states of system (7) is linearly bounded, here these two stabilities are assumed to be satisfied with quadratic gain functions as in [14].

(iii) From Assumptions A1 and B2, system (7) is assumed to be dominated by a general triangular system with linear growth nonlinear terms or bounded diffusion terms. It should be pointed that this class of systems represents an important class of stochastic nonlinear systems which are not covered by the previous work.

4. HIGH-GAIN OBSERVER DESIGN

First, we introduce a state-estimator for subsystem (7):

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + La_1(y-\hat{x}_1)\\ &\vdots\\ \dot{\hat{x}}_{n-1} &= \hat{x}_n + L^{n-1}a_{n-1}(y-\hat{x}_1)\\ \dot{\hat{x}}_n &= u + L^n a_n(y-\hat{x}_1)\end{aligned} \tag{11}$$

where $L\geqslant 1$ is a gain parameter to be determined later, and $a_i>0$, $i=1,\dots,n$, are real numbers such that the polynomial $p(s)=s^n+a_1s^{n-1}+\cdots+a_n$ is Hurwitz.

Let $\varepsilon_i=(x_i-\hat{x}_i)/L^{i-1}, i=1,\dots,n$ and $\varepsilon=[\varepsilon_1,\dots,\varepsilon_n]^{\mathrm{T}}$. Then we obtain the following error dynamics:

$$\mathrm{d}\varepsilon = [LA\varepsilon + F_\varepsilon]\,\mathrm{d}t + G_\varepsilon\Sigma\,\mathrm{d}w \tag{12}$$

where

$$A=\begin{bmatrix} -a_1 & & \\ \vdots & & I_{n-1} \\ -a_{n-1} & & \\ -a_n & 0 & \cdots\ 0\end{bmatrix},\quad F_\varepsilon=\begin{bmatrix} f_1\\ \frac{1}{L}f_2\\ \vdots\\ \frac{1}{L^{n-1}}f_n\end{bmatrix},\quad G_\varepsilon=\begin{bmatrix} g_1\\ \frac{1}{L}g_2\\ \vdots\\ \frac{1}{L^{n-1}}g_n\end{bmatrix}$$

For the polynomial $p(s)=s^n+a_1s^{n-1}+\cdots+a_n$ is designed to be Hurwitz, there exists a positive-definite matrix P such that

$$A^{\mathrm{T}}P+PA=-I$$

In the following, we give error dynamics analysis for the two problems mentioned above.

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4.1. Error dynamics analysis of disturbance attenuation problem

By Assumption A1 and $L\geqslant 1$, we have the following estimates:

$$\begin{aligned}|F_\varepsilon|^2 &= \sum_{i=1}^{n}\left|\frac{1}{L^{i-1}}f_i\right|^2 \leqslant \sum_{i=1}^{n}\left[\frac{1}{L^{i-1}}C_f(|x_1|+\cdots+|x_i|)+K_f|x_z|\right]^2\\ &\leqslant 2\sum_{i=1}^{n}\left[\frac{1}{L^{i-1}}C_f(|x_1|+\cdots+|x_i|)\right]^2+2nK_f^2|x_z|^2\\ &\leqslant 2nC_f^2\left(|x_1|+\frac{|x_2|}{L}+\cdots+\frac{|x_n|}{L^{n-1}}\right)^2+2nK_f^2|x_z|^2\end{aligned}\tag{13}$$

$$|G_\varepsilon|^2=\sum_{i=1}^{n}\left|\frac{1}{L^{i-1}}g_i\right|^2\leqslant nC_g^2\tag{14}$$

Let $V_e(\varepsilon)=\delta\varepsilon^{\mathrm{T}}P\varepsilon$, where $\delta>0$ is a parameter to be specified later. Then, by (12)–(14) and Itô formula, we obtain

$$\begin{aligned}\mathscr{L}V_e &= \delta L\varepsilon^{\mathrm{T}}(A^{\mathrm{T}}P+PA)\varepsilon+2\delta\varepsilon^{\mathrm{T}}PF_\varepsilon+\delta\,\mathrm{Tr}(\Sigma(t)^{\mathrm{T}}G_\varepsilon^{\mathrm{T}}PG_\varepsilon\Sigma(t))\\ &\leqslant -\delta L|\varepsilon|^2+\delta[|\varepsilon^{\mathrm{T}}P|^2+|F_\varepsilon|^2]+\delta\lambda_{\max}(P)|G_\varepsilon|^2||\Sigma(t)||^2\\ &\leqslant -\delta L|\varepsilon|^2+\delta||P||^2|\varepsilon|^2+2n\delta C_f^2\left(|x_1|+\frac{|x_2|}{L}+\cdots+\frac{|x_n|}{L^{n-1}}\right)^2+\delta 2nK_f^2|x_z|^2\\ &\quad+\delta\lambda_{\max}(P)nC_g^2||\Sigma(t)||^2\\ &= -\delta(L-||P||^2)|\varepsilon|^2+2n\delta C_f^2\left(|x_1|+\frac{|x_2|}{L}+\cdots+\frac{|x_n|}{L^{n-1}}\right)^2\\ &\quad+2n\delta K_f^2|x_z|^2+\delta\lambda_{\max}(P)nC_g^2||\Sigma(t)||^2\end{aligned}$$

Noticing $|(1/L^{i-1})x_i|\leqslant|(1/L^{i-1})\hat{x}_i|+|\varepsilon_i|$, we have

$$\begin{aligned}\mathscr{L}V_e &\leqslant -\delta(L-||P||^2)|\varepsilon|^2+C_{e1}\left(|\varepsilon_1|+\cdots+|\varepsilon_n|+|\hat{x}_1|+\frac{|\hat{x}_2|}{L}+\cdots+\frac{|\hat{x}_n|}{L^{n-1}}\right)^2+\Delta_1(x_z,\Sigma)\\ &\leqslant -\delta(L-||P||^2)|\varepsilon|^2+2C_{e1}(|\varepsilon_1|+\cdots+|\varepsilon_n|)^2+2C_{e1}\left(|\hat{x}_1|+\frac{|\hat{x}_2|}{L}+\cdots+\frac{|\hat{x}_n|}{L^{n-1}}\right)^2+\Delta_1(x_z,\Sigma)\\ &\leqslant -[\delta(L-||P||^2)-2nC_{e1}]|\varepsilon|^2+2C_{e1}n\left(|\hat{x}_1|^2+\frac{|\hat{x}_2|^2}{L^2}+\cdots+\frac{|\hat{x}_n|^2}{L^{2n-2}}\right)+\Delta_1(x_z,\Sigma)\end{aligned}\tag{15}$$

where $C_{e1}=2n\delta C_f^2$, $\Delta_1(x_z,\Sigma)=2n\delta K_f^2|x_z|^2+\delta\lambda_{\max}(P)nC_g^2||\Sigma||^2$.

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4.2. Error dynamics analysis of asymptotic stabilization problem

By Assumptions B1 and B2 and $L \geqslant 1$, we have the following estimates:

$$|G_\varepsilon|^2 \leqslant 2nC_g^2 \left(|x_1| + \frac{|x_2|}{L} + \cdots + \frac{|x_n|}{L^{n-1}} \right)^2 + 2nK_g^2 |x_z|^2 \tag{16}$$

Similar to the above subsection, let $V_e(\varepsilon) = \delta \varepsilon^{\mathrm{T}} P \varepsilon$. Then, by (12), (13), (16) and $|(1/L^{i-1}) x_i| \leqslant |(1/L^{i-1}) \hat{x}_i| + |\varepsilon_i|$, we obtain

$$\mathscr{L} V_e \leqslant -[\delta(L - ||P||^2) - 2nC_{e2}]|\varepsilon|^2 + 2nC_{e2} \left(|\hat{x}_1|^2 + \frac{|\hat{x}_2|^2}{L^2} + \cdots + \frac{|\hat{x}_n|^2}{L^{2n-2}} \right) + \Delta_2(x_z) \tag{17}$$

where $C_{e2} = 2n\delta(C_f^2 + \lambda_{\max}(P) C_g^2)$, $\Delta_2(x_z) = 2n\delta(K_f^2 + \lambda_{\max}(P) K_g^2)|x_z|^2$.

Remark 5

Here, similar to the deterministic case, we design a high-gain observer (11) with a to-be-determined gain parameter L for partially unmeasurable states. But due to the existence of the noise, the error dynamic analysis is more complex than the deterministic case. In the next section, we will design the linear output-feedback controller and gain parameter L simultaneously.

5. OUTPUT-FEEDBACK CONTROLLER DESIGN

In this section, we supply a unified control design procedure for the disturbance attenuation and asymptotic stabilization problems by using feedback domination design method. In the sequel, C_e will be used to denote C_{e1} or C_{e2}, and Δ denotes Δ_1 or Δ_2, accordingly.

Step 1: Let $z_1 = \hat{x}_1$ and $V_1 = V_e + \frac{1}{2}\hat{x}_1^2$. Then, by (11) and (15) or (17), we have

$$\begin{aligned} \mathscr{L} V_1 \leqslant & -[\delta(L - ||P||^2) - 2nC_e]|\varepsilon|^2 + 2nC_e \left(|\hat{x}_1|^2 + \frac{|\hat{x}_2|^2}{L^2} + \cdots + \frac{|\hat{x}_n|^2}{L^{2n-2}} \right) \\ & + \Delta + \hat{x}_1(\hat{x}_2 + L a_1 \varepsilon_1) \end{aligned} \tag{18}$$

Defining $z_2 = \hat{x}_2 - \phi_1(\hat{x}_1)$ and noticing that

$$\hat{x}_1 L a_1 \varepsilon_1 \leqslant L \frac{a_1^2 |\hat{x}_1|^2}{4} + L \varepsilon_1^2$$

$$2nC_e |\hat{x}_1|^2 \leqslant 2nC_e L |\hat{x}_1|^2, \quad \hat{x}_1 z_2 \leqslant \frac{1}{4L} z_2^2 + L |\hat{x}_1|^2$$

$$2nC_e \frac{|\hat{x}_2|^2}{L^2} = 2nC_e \frac{|z_2 + \phi_1(\hat{x}_1)|^2}{L^2} \leqslant 4nC_e \frac{z_2^2 + \phi_1^2}{L^2}$$

by (18) we have

$$\begin{aligned} \mathscr{L} V_1 \leqslant & -[\delta(L - ||P||^2) - 2nC_e - L]|\varepsilon|^2 + 2nC_e \left(\frac{|\hat{x}_3|^2}{L^4} + \frac{|\hat{x}_4|^2}{L^6} + \cdots + \frac{|\hat{x}_n|^2}{L^{2n-2}} \right) \\ & + \hat{x}_1 \left(\phi_1 + 2nC_e L \hat{x}_1 + \frac{L a_1^2}{4} \hat{x}_1 + L \hat{x}_1 \right) + 4nC_e \frac{z_2^2}{L^2} + 4nC_e \frac{\phi_1^2}{L^2} + \frac{1}{4L} z_2^2 + \Delta \end{aligned} \tag{19}$$

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Different from the control design of the systems in output-feedback form, here the virtual control ϕ_1 appears as a disturbance in the term $4nC_e\phi_1^2/L^2$, which can be canceled by properly designing the parameter L.

Define the virtual control law

$$\phi_1(\hat{x}_1) = -Lb_1\hat{x}_1 \tag{20}$$

where $b_1 = 2n - 1 + 2nC_e + a_1^2/4 + 1$. Then it follows from (19) that

$$\begin{aligned}\mathscr{L}V_1 \leqslant & -[\delta(L - ||P||^2) - 2nC_e - L]|\varepsilon|^2 + 2nC_e\left(\frac{|\hat{x}_3|^2}{L^4} + \frac{|\hat{x}_4|^2}{L^6} + \cdots + \frac{|\hat{x}_n|^2}{L^{2n-2}}\right)\\ & -(2n-1)L\hat{x}_1^2 + \frac{4nC_e}{L^2}z_2^2 + 4nC_eb_1^2\hat{x}_1^2 + \frac{1}{4L}z_2^2 + \Delta\\ = & -[\delta(L - ||P||^2) - 2nC_e - L]|\varepsilon|^2 + 2nC_e\left(\frac{|\hat{x}_3|^2}{L^4} + \frac{|\hat{x}_4|^2}{L^6} + \cdots + \frac{|\hat{x}_n|^2}{L^{2n-2}}\right)\\ & -[2nL - 4nC_eb_1^2 - L]|\hat{x}_1|^2 + \frac{4nC_e}{L^2}z_2^2 + \frac{1}{4L}z_2^2 + \Delta\end{aligned} \tag{21}$$

Step k ($k = 2, \ldots, n-1$): At this step, we can obtain a property similar to (21), which is presented by the following lemma.

Lemma 1

For every $k = 1, \ldots, n-1$, there exist smooth functions $\phi_i, (1 \leqslant i \leqslant k)$ such that $\phi_i(0) = 0$ and along the solutions of (11) and (12), the Lyapunov function candidate $V_k = V_1 + \sum_{i=2}^{k}(1/2L^{2(i-1)})z_i^2$ satisfies

$$\begin{aligned}\mathscr{L}V_k \leqslant & -[\delta(L - ||P||^2) - 2nC_e - kL]|\varepsilon|^2 + 2nC_e\left(\frac{|\hat{x}_{k+2}|^2}{L^{2k+2}} + \cdots + \frac{|\hat{x}_n|^2}{2L^{2n-2}}\right)\\ & -(2nL - 4nC_eb_1^2 - kL)z_1^2 - \sum_{j=2}^{k}\frac{1}{L^{2j-2}}[(n+j-k)L - 4nC_eb_j^2]z_j^2\\ & + \frac{4nC_e}{L^{2k}}z_{k+1}^2 + \frac{1}{4L^{2k-1}}z_{k+1}^2 + \Delta\end{aligned} \tag{22}$$

where

$$z_i = \hat{x}_i - \phi_{i-1}(\overline{\hat{x}}_{i-1}), \quad i = 2, \ldots, k, \quad \hat{x}_{n+1} = u, \ \ z_{n+1} = 0$$

$$\phi_{i-1} = -Lb_{i-1}z_{i-1}$$

$$z_{i-1} = \hat{x}_{i-1} + Lb_{i-2}\hat{x}_{i-2} + L^2b_{i-2}b_{i-3}\hat{x}_{i-3} + \cdots + L^{i-2}b_{i-2}b_{i-3}\cdots b_1\hat{x}_1$$

$$b_i = n + 4nC_e + \frac{1}{4} + \frac{\tilde{d}_i^2}{4} + \frac{d_{i1}^2}{4} + \cdots + \frac{d_{i,i-1}^2}{4} + d_{ii} + 1, \quad i = 2, \ldots, n-1$$

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$$\tilde{d}_i = a_i + b_{i-1} \cdots b_1 a_1 + b_{i-1} \cdots b_2 a_2 + \cdots + b_{i-1} a_{i-1}$$
$$d_{i1} = -b_{i-1} \cdots b_1 b_1$$
$$d_{ij} = b_{i-1} \cdots b_j b_{j-1} - b_{i-1} \cdots b_j b_j, \quad 2 \leqslant j \leqslant i-1$$
$$d_{ii} = b_{i-1}$$

Proof
See Appendix B. □

Using the inductive argument step by step, at the nth step, by the foot note appearing in Appendix B on page 21, one can design the control law

$$\begin{aligned} u &= -Lb_n z_n = -Lb_n(\hat{x}_n + Lb_{n-1}(\hat{x}_{n-1} + \cdots + Lb_2(\hat{x}_2 + Lb_1\hat{x}_1)\cdots)) \\ &= -Lb_n\hat{x}_n - L^2 b_n b_{n-1}\hat{x}_{n-1} - \cdots - L^n b_n b_{n-1} b_{n-2} \cdots b_1 \hat{x}_1 \end{aligned} \tag{23}$$

where

$$b_n = n + 4nC_e + \frac{1}{4} + \frac{\tilde{d}_n^2}{4} + \frac{d_{n1}^2}{4} + \cdots + \frac{d_{n,n-1}^2}{4} + d_{nn}$$

In this case, we have

$$\begin{aligned} \mathscr{L}V_n \leqslant & -[\delta(L - ||P||^2) - 2nC_e - nL]|\varepsilon|^2 - (nL - 4nC_e b_1^2) z_1^2 \\ & - \sum_{j=2}^{n-1} \frac{1}{L^{2j-2}}[(j+1)L - 4nC_e b_j^2] z_j^2 - nL^{3-2n} z_n^2 \end{aligned} \tag{24}$$

6. STABILITY ANALYSIS OF THE CLOSED-LOOP SYSTEM

6.1. Disturbance attenuation problem

By (24) and the definitions of C_e and Δ (with respect to Assumption A1), we have

$$\begin{aligned} \mathscr{L}V_n \leqslant & -[\delta(L - ||P||^2) - 2nC_{e1} - nL]|\varepsilon|^2 + 2n\delta K_f^2|x_z|^2 + \delta\lambda_{\max}(P)nC_g^2||\Sigma(t)||^2 \\ & - (nL - 4nC_{e1}b_1^2)z_1^2 - \sum_{j=2}^{n-1} \frac{1}{L^{2j-2}}[(j+1)L - 4nC_e b_j^2] z_j^2 - nL^{3-2n} z_n^2 \end{aligned} \tag{25}$$

Consider the following Lyapunov function:

$$W_1(x_z, \bar{x}_n) = V_n + \frac{q_1 + \epsilon}{\alpha} V_z$$

where $q_1 = 2n\delta K_f^2, \epsilon > 0$. Then, by (8), (25) and $|y|^2 = |\hat{x}_1 + \varepsilon_1|^2 \leqslant 2|\hat{x}_1|^2 + 2|\varepsilon_1|^2$, we have

$$\begin{aligned} \mathscr{L}W_1 \leqslant & -c_1|\varepsilon|^2 - c_2 z_1^2 - \sum_{j=2}^{n} \lambda_j z_j^2 + p||\Sigma||^2 - \epsilon|x_z|^2 + \frac{q_1+\epsilon}{\alpha}\gamma|y|^2 \\ \leqslant & -c_1|\varepsilon|^2 - c_2 z_1^2 - \sum_{j=2}^{n} \lambda_j z_j^2 - \epsilon|x_z|^2 + 2\frac{q_1+\epsilon}{\alpha}\gamma z_1^2 + 2\frac{q_1+\epsilon}{\alpha}\gamma|\varepsilon|^2 + p||\Sigma||^2 \end{aligned} \tag{26}$$

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where

$$
\begin{aligned}
c_1 &= \delta(L - ||P||^2) - 2nC_{e1} - nL \\
c_2 &= nL - 4nC_{e1} b_1^2 \\
\lambda_j &= \frac{1}{L^{2j-2}}[jL - 4nC_{e1} b_j^2], \quad j = 2, \ldots, n-1 \\
\lambda_n &= nL^{3-2n} \\
C_{e1} &= 2n\delta C_f^2 \\
p &= \delta\lambda_{\max}(P) n C_g^2 + \frac{q_1 + \epsilon}{\alpha}\gamma_w
\end{aligned}
$$

Choose the design parameters $L \geqslant 1$, $\delta > 0$ and $\epsilon > 0$ such that

$$
\begin{aligned}
c_{11} &= c_1 - 2\frac{q_1 + \epsilon}{\alpha}\gamma > 0 \\
\lambda_1 &= c_2 - 2\frac{q_1 + \epsilon}{\alpha}\gamma > 0 \\
\lambda_j &= \frac{1}{L^{2j-2}}[jL - 4nC_{e1} b_j^2]0
\end{aligned}
$$

Then, it follows from (26) that

$$
\mathscr{L} W_1 \leqslant -c_{11}|\varepsilon|^2 - \sum_{j=1}^{n} \lambda_j z_j^2 - \epsilon|x_z|^2 + p||\Sigma||^2
$$

Noticing that $(1/\sqrt{r})||\Sigma||^2 \leqslant ||\Sigma\Sigma^{\mathrm{T}}|| \leqslant ||\Sigma||^2$, by Theorem 2, we have the following result.

Theorem 3
For system (6)–(7), suppose Assumptions A1 and A2 hold. Then under control law (23), the closed-loop system is stochastic disturbance attenuation in the NSS sense.

6.2. Asymptotic stabilization problem

By (24) and the definitions of C_e and Δ (with respect to Assumptions B1 and B2), we have

$$
\begin{aligned}
\mathscr{L} V_n \leqslant & -[\delta(L - ||P||^2) - 2nC_{e2} - nL]|\varepsilon|^2 + 2n\delta(K_f^2 + \lambda_{\max}(P)K_g^2)|x_z|^2 \\
& -(nL - 4nC_{e2} b_1^2) z_1^2 - \sum_{j=2}^{n-1} \frac{1}{L^{2j-2}}[(j+1)L - 4nC_e b_j^2] z_j^2 - nL^{3-2n} z_n^2
\end{aligned} \tag{27}
$$

Consider the following Lyapunov function:

$$
W_2(x_z, \bar{x}_n) = V_n + \frac{q_2 + \epsilon}{\alpha} V_z
$$

where $q_2 = 2n\delta(K_f^2 + \lambda_{\max}(P)K_g^2), \epsilon > 0$. Then, similar to (26), by (9) and (27) we have

$$
\begin{aligned}
\mathscr{L} W_2 \leqslant & -c_1|\varepsilon|^2 - c_2 z_1^2 - \sum_{j=2}^{n} \lambda_j z_j^2 - \epsilon|x_z|^2 + \frac{q_2 + \epsilon}{\alpha}\gamma|y|^2 \\
\leqslant & -c_1|\varepsilon|^2 - c_2 z_1^2 - \sum_{j=2}^{n} \lambda_j z_j^2 - \epsilon|x_z|^2 + 2\frac{q_2 + \epsilon}{\alpha}\gamma z_1^2 + 2\frac{q_2 + \epsilon}{\alpha}\gamma|\varepsilon|^2
\end{aligned} \tag{28}
$$

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where

$$
\begin{aligned}
c_1 &= \delta(L - ||P||^2) - 2nC_{e2} - nL \\
c_2 &= nL - 4nC_{e2}b_1^2 \\
\lambda_j &= \frac{1}{L^{2j-2}}[jL - 4nC_{e2}b_j^2], \quad j = 2, \dots, n-1 \\
\lambda_n &= nL^{3-2n} \\
C_{e2} &= 2n\delta(C_f^2 + \lambda_{\max}(P)C_g^2)
\end{aligned}
$$

Choose the design parameters $L \geqslant 1$, $\delta > 0$ and $\epsilon > 0$ such that

$$
\begin{aligned}
c_{11} &= c_1 - 2\frac{q_2 + \epsilon}{\alpha}\gamma > 0 \\
\lambda_j &= \frac{1}{L^{2j-2}}[jL - 4nC_{e2}b_j^2] > 0 \\
\lambda_1 &= c_2 - 2\frac{q_2 + \epsilon}{\alpha}\gamma > 0
\end{aligned}
$$

Then, by (28) we have

$$
\mathscr{L}W_2 \leqslant -c_{11}|\varepsilon|^2 - \sum_{j=1}^{n} \lambda_j z_j^2 - \epsilon |x_z|^2
$$

Thus, by Theorem 1 we have the following results.

Theorem 4
For system (6)–(7), suppose Assumptions B1–B3 hold. Then under control law (23), the closed-loop system has an almost surely unique solution on $[0, \infty)$, and its zero solution is GASiP.

Remark 6
Different from the previous work about output-feedback control for stochastic nonlinear systems, in this paper, a quadratic Lyapunov function is adopted instead of a locally quadratic or quartic function, which simplifies the controller design greatly.

Remark 7
Unlike linear time-invariant systems, here separation principle does not hold due to essential nonlinearity of the system, and the controller design is more difficult than that in linear cases.

7. CONCLUSION

In this paper, the problems of stochastic disturbance attenuation and asymptotic stabilization *via* output feedback have been studied for a class of stochastic nonlinear systems with linearly bounded unmeasurable states. The methodology previously developed for deterministic systems has been generalized to stochastic ones. Different from the previous work about output-feedback control for stochastic nonlinear systems, here a quadratic Lyapunov function was

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adopted instead of a locally quadratic or quartic function. Under the assumption that the inverse dynamics are GSISS, a linear output-feedback controller was explicitly constructed to make the closed-loop system noise-to-state stable. When the intensity of noise is known to be a unit matrix, under the assumption that the inverse dynamics are SISS, a linear output-feedback controller was explicitly constructed to make the closed-loop system globally asymptotically stable in probability.

APPENDIX A: PROOF OF THEOREM 2

Conclusion (b) comes from conclusion (a) and the definition of NSS [19] directly. So, here it suffices to show conclusion (a).

Let

$$\mathscr{B}=\left\{x:|x|<\alpha^{-1}(d\chi(\|v\|_\infty)+d\chi_w\left(\sup_{t\geqslant 0}\|\Sigma(t)\Sigma(t)^{\mathrm{T}}\|)\right)\right\},\quad \mathscr{B}^c=\mathbb{R}^n\backslash\mathscr{B}$$

where $\|v\|_\infty=\sup_{t\geqslant 0}\|v_t\|=\sup_{t\geqslant 0}\inf_{\mathscr{A}\subset\Omega,P(\mathscr{A})=0}\sup\{|v(x(\omega,t),t)|:\omega\in\Omega\backslash\mathscr{A}\}$, and $d\geqslant 1$ is a constant. Define a sequence of stopping times $\{\tau_i\}_{i\geqslant 0}$:

$$\tau_0=0$$

$$\tau_1=\begin{cases}\inf\{t>\tau_0:x(t)\in\mathscr{B}\} & \text{if } \{t>\tau_0:x(t)\in\mathscr{B}\}\neq\emptyset\\ \infty & \text{otherwise}\end{cases}$$

$$\tau_{2i}=\begin{cases}\inf\{t>\tau_{2i-1}:x(t)\in\mathscr{B}^c\} & \text{if } \{t>\tau_{2i-1}:x(t)\in\mathscr{B}^c\}\neq\emptyset\\ \infty & \text{otherwise}\end{cases}$$

$$\tau_{2i+1}=\begin{cases}\inf\{t>\tau_{2i}:x(t)\in\mathscr{B}\} & \text{if } \{t>\tau_{2i}:x(t)\in\mathscr{B}\}\neq\emptyset\\ \infty & \text{otherwise}\end{cases}$$

where $i=1,2,\ldots$. Noticing that $\mathscr{B}^c$ is a closed set, for any $t\geqslant 0$ and any $i=1,2,\ldots,$ if $t\in[\tau_{2i},\tau_{2i+1}]$, then $x(t)\in\mathscr{B}^c$; and if $t\in(\tau_{2i+1},\tau_{2i+2})$, then $x(t)\in\mathscr{B}$.

We now complete the proof by considering the following two cases: $x_0\in\mathscr{B}^c\backslash\{0\}$ and $x_0\in\mathscr{B}\backslash\{0\}$, respectively.

Case 1: $x_0\in\mathscr{B}^c\backslash\{0\}$. In this case, for any $t\in[0,\tau_1]$, $x(t)\in\mathscr{B}^c$.

By the definitions of τ_{2i} and τ_{2i+1}, for any $t\in[\tau_{2i},\tau_{2i+1}]$, $i=0,1,2,\ldots,$

$$|x(t)|\geqslant\alpha^{-1}\left(d\chi(\|v\|_\infty)+d\chi_w\left(\sup_{t\geqslant 0}\|\Sigma(t)\Sigma(t)^{\mathrm{T}}\|\right)\right)\geqslant\alpha^{-1}\left(d\chi(|v|)+d\chi_w\left(\sup_{t\geqslant 0}\|\Sigma(t)\Sigma(t)^{\mathrm{T}}\|\right)\right)\quad\text{a.s.}$$

which together with (5) leads to

$$\mathscr{L}V(x(t),t)\leqslant-\left(1-\frac{1}{d}\right)\alpha(|x(t)|)\quad\text{a.s.}\tag{A1}$$

By (2) and Itô formula, we have

$$V(x(t),t)=V(x(0),0)+\int_0^t\mathscr{L}V(x(s),s)\,\mathrm{d}s+\int_0^t\frac{\partial V(x(s),s)}{\partial x}g(x(s),v(x(s),s),s)\Sigma(s)\,\mathrm{d}w(s)\tag{A2}$$

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and by [27, p. 72], for any $t\geqslant 0$, $i=0,1,2,\ldots$,

$$V(x(t\wedge\tau_{2i}),t\wedge\tau_{2i})=V(x(0),0)+\int_0^{t\wedge\tau_{2i}}\mathscr{L}V(x(s),s)\,\mathrm{d}s+\int_0^{t\wedge\tau_{2i}}\frac{\partial V(x(s),s)}{\partial x}g(x(s),v(x(s),s),s)\Sigma(s)\,\mathrm{d}w(s)$$

$$V(x(t\wedge\tau_{2i+1}),t\wedge\tau_{2i+1})=V(x(0),0)+\int_0^{t\wedge\tau_{2i+1}}\mathscr{L}V(x(s),s)\,\mathrm{d}s+\int_0^{t\wedge\tau_{2i+1}}\frac{\partial V(x(s),s)}{\partial x}g(x(s),v(x(s),s),s)\Sigma(s)\,\mathrm{d}w(s)$$

From the above two equalities and Lemma 4.1 of Chapter 4 in [27], it follows that

$$\begin{aligned}&V(x(t\wedge\tau_{2i+1}),t\wedge\tau_{2i+1})-V(x(t\wedge\tau_{2i}),t\wedge\tau_{2i})\\&=\int_{t\wedge\tau_{2i}}^{t\wedge\tau_{2i+1}}\mathscr{L}V(x(s),s)\,\mathrm{d}s+\int_{t\wedge\tau_{2i}}^{t\wedge\tau_{2i+1}}\frac{\partial V(x(s),s)}{\partial x}g(x(s),v(x(s),s),s)\Sigma(s)\,\mathrm{d}w(s)\end{aligned}\tag{A3}$$

By Lemma 4.1 and Theorem 4.7 of Chapter 4 in [27], we obtain that

$$\begin{aligned}&\int_{t\wedge\tau_{2i}}^{t\wedge\tau_{2i+1}}\frac{\partial V(x(s),s)}{\partial x}g(x(s),v(x(s),s),s)\Sigma(s)\,\mathrm{d}w(s)\\&=\int_{\tau_{2i}}^{(t\vee\tau_{2i})\wedge\tau_{2i+1}}\frac{\partial V(x(s),s)}{\partial x}g(x(s),v(x(s),s),s)\Sigma(s)\,\mathrm{d}w(s)\quad\text{a.s.}\end{aligned}\tag{A4}$$

Noticing that

$$\begin{aligned}&V(x(t\wedge\tau_{2i+1}),t\wedge\tau_{2i+1})-V(x(t\wedge\tau_{2i}),t\wedge\tau_{2i})\\&=V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}),(t\vee\tau_{2i})\wedge\tau_{2i+1})-V(x(\tau_{2i}),\tau_{2i})\end{aligned}$$

and

$$\int_{t\wedge\tau_{2i}}^{t\wedge\tau_{2i+1}}\mathscr{L}V(x(s),s)\,\mathrm{d}s=\int_{\tau_{2i}}^{(t\vee\tau_{2i})\wedge\tau_{2i+1}}\mathscr{L}V(x(s),s)\,\mathrm{d}s$$

by (A3) and (A4), we have

$$\begin{aligned}V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}),(t\vee\tau_{2i})\wedge\tau_{2i+1})&=V(x(\tau_{2i}),\tau_{2i})+\int_{\tau_{2i}}^{(t\vee\tau_{2i})\wedge\tau_{2i+1}}\mathscr{L}V(x(s),s)\,\mathrm{d}s\\&\quad+\int_{\tau_{2i}}^{(t\vee\tau_{2i})\wedge\tau_{2i+1}}\frac{\partial V(x(s),s)}{\partial x}g(x(s),v(x(s),s),s)\Sigma(s)\,\mathrm{d}w(s)\quad\text{a.s.}\end{aligned}$$

According to the above equality and (A1), noticing that $d\geqslant 1$, we obtain that the process $V_t^i:=V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}),(t\vee\tau_{2i})\wedge\tau_{2i+1}))$ is a supermartingale.

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Thus, by [19, Theorem 3.3], for any $\epsilon' \in (0,1)$, there exists a class $\mathscr{K}\mathscr{L}$-function $\beta_i(\cdot,\cdot)$ such that

$$P\{|x((t\vee\tau_{2i})\wedge\tau_{2i+1})|<\beta_i(|x_{\tau_{2i}}|,t)\}\geqslant 1-\epsilon' \quad \forall t\geqslant 0,\ \ x(\tau_{2i})\in\mathbb{R}^n\backslash\{0\}$$

In particular, for $i=0$, if we write β_0 as β, then

$$P\{|x(t\wedge\tau_1)|<\beta(|x_0|,t)\}\geqslant 1-\epsilon' \quad \forall t\geqslant 0,\ \ x_0\in\mathbb{R}^n\backslash\{0\} \tag{A5}$$

Now let us pay attention to $x(t\vee\tau_1)$. Define

$$\mathscr{A}=\bigcup_{i=0}^{\infty}(\tau_{2i+1},\tau_{2i+2}),\quad \mathscr{C}=\bigcup_{i=1}^{\infty}[\tau_{2i},\tau_{2i+1}]$$

Then, $\mathscr{A}\cap\mathscr{C}=\emptyset$ and $(\tau_1,\infty)=\mathscr{A}\cup\mathscr{C}$, and hence,

$$\begin{aligned}
E[V(x(t\vee\tau_1),t\vee\tau_1)] &= E[V(x(t\vee\tau_1),t\vee\tau_1)\cdot I_{\{t\in[0,\tau_1]\}}]\\
&\quad + E[V(x(t\vee\tau_1),t\vee\tau_1)\cdot I_{\{t\in(\tau_1,\infty)\}}]\\
&= E[V(x(\tau_1),\tau_1)\cdot I_{\{t\in[0,\tau_1]\}}]+E[V(x(t\vee\tau_1),t\vee\tau_1)\cdot I_{\{t\in\mathscr{A}\}}]\\
&\quad + E[V(x(t\vee\tau_1),t\vee\tau_1)\cdot I_{\{t\in\mathscr{C}\}}]\\
&= E[V(x(\tau_1),\tau_1)\cdot I_{\{t\in[0,\tau_1]\}}]+\sum_{i=0}^{\infty}E[V(x(t),t)\cdot I_{\{t\in(\tau_{2i+1},\tau_{2i+2})\}}]\\
&\quad +\sum_{i=1}^{\infty}E[V(x(t),t)\cdot I_{\{t\in[\tau_{2i},\tau_{2i+1}]\}}]
\end{aligned} \tag{A6}$$

Since $V_t^i := V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}),(t\vee\tau_{2i})\wedge\tau_{2i+1})$ is a supermartingale, we have

$$E[V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}),(t\vee\tau_{2i})\wedge\tau_{2i+1})]\leqslant E[V(x(\tau_{2i}),\tau_{2i})] \tag{A7}$$

By the continuity of the trajectory, $x(\tau_{2i})$ and $x(\tau_{2i+1})$ lie on the boundary of the set $\mathscr{B}$, i.e. $x(\tau_{2i})=x(\tau_{2i+1})=\alpha^{-1}(d\chi(||v||_\infty+d\chi_w(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||))$ is a constant. Hence, we have

$$E[V(x(\tau_1),\tau_1)\cdot I_{\{t\in[0,\tau_1]\}}]\leqslant P\{t\in[0,\tau_1]\}\left[\alpha_2\left(\alpha^{-1}\left(d\chi(||v||_\infty)+d\chi_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right)\right)\right] \tag{A8}$$

and by (A7) and (4),

$$\begin{aligned}
E[V(x(t),t)\cdot I_{\{t\in[\tau_{2i},\tau_{2i+1}]\}}] &= E[V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}),(t\vee\tau_{2i})\wedge\tau_{2i+1})\cdot I_{\{t\in[\tau_{2i},\tau_{2i+1}]\}}]\\
&= E[V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}),(t\vee\tau_{2i})\wedge\tau_{2i+1})]\\
&\quad - E[V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}),(t\vee\tau_{2i})\wedge\tau_{2i+1})\cdot I_{\{t<\tau_{2i}\}}]\\
&\quad - E[V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}),(t\vee\tau_{2i})\wedge\tau_{2i+1})\cdot I_{\{t>\tau_{2i+1}\}}]\\
&= E[V(x((t\vee\tau_{2i})\wedge\tau_{2i+1}),(t\vee\tau_{2i})\wedge\tau_{2i+1})]\\
&\quad - E[V(x(\tau_{2i}),\tau_{2i})\cdot I_{\{t<\tau_{2i}\}}]\\
&\quad - E[V(x(\tau_{2i+1}),\tau_{2i+1})\cdot I_{\{t>\tau_{2i+1}\}}]
\end{aligned} \tag{A9}$$

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$$
\begin{aligned}
&\leqslant E[V(x(\tau_{2i}),\tau_{2i})]-\alpha_1(|x(\tau_{2i}|)\cdot P\{t<\tau_{2i}\}\\
&\quad -\alpha_1(|x(\tau_{2i+1}|)\cdot P\{t>\tau_{2i+1}\}\\
&\leqslant \alpha_2\left(\alpha^{-1}\left(d\chi(||v||_\infty)+d\chi_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right)\right)\\
&\quad -\alpha_1\left(\alpha^{-1}\left(d\chi(||v||_\infty)+d\chi_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right)\right)\\
&\quad \cdot P\{t<\tau_{2i}\}\cup\{t>\tau_{2i+1}\}\\
&=P\{t\in[\tau_{2i},\tau_{2i+1}]\}\cdot\alpha_1\left(\alpha^{-1}\left(d\chi(||v||_\infty)+d\chi_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right)\right)\\
&\quad +\alpha_2\left(\alpha^{-1}\left(d\chi(||v||_\infty)+d\chi_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right)\right)\\
&\quad -\alpha_1\left(\alpha^{-1}\left(d\chi(||v||_\infty)+d\chi_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right)\right)\\
&\leqslant P\{t\in[\tau_{2i},\tau_{2i+1}]\}\cdot\alpha_2\left(\alpha^{-1}\left(d\chi(||v||_\infty)+d\chi_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right)\right)\\
&\quad +\alpha_2\left(\alpha^{-1}\left(d\chi(||v||_\infty)+d\chi_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right)\right)
\end{aligned}
$$

Noticing that $t\in(\tau_{2i+1},\tau_{2i+2})$ implies $x(t)\in\mathscr{B}$, we have

$$
\begin{aligned}
&\sum_{i=0}^{\infty}E[V(x(t),t)I_{\{t\in(\tau_{2i+1},\tau_{2i+2})\}}]\\
&\quad\leqslant\sum_{i=0}^{\infty}P\{t\in(\tau_{2i+1},\tau_{2i+2})\}\cdot\left[\alpha_2\left(\alpha^{-1}\left(d\chi(||v||_\infty)+d\chi_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right)\right)\right]
\end{aligned}\tag{A10}
$$

Thus, by (A6), (A8)–(A10) one can obtain

$$
E[V(x(t\vee\tau_1),t\vee\tau_1)]\leqslant 2\alpha_2\left(\alpha^{-1}\left(d\chi(||v||_\infty)+d\chi_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right)\right)\tag{A11}
$$

Recalling that $V(x,t)$ is nonnegative, we have

$$
\begin{aligned}
&E[V(x(t\vee\tau_1),t\vee\tau_1)]\\
&\quad\geqslant E[V(x(t\vee\tau_1),t\vee\tau_1)\cdot I_{\{V(x(t\vee\tau_1),t\vee\tau_1)\geqslant\delta(\alpha_2(\alpha^{-1}(d\chi(||v||_\infty)+d\chi_w(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||))))\}}]\\
&\quad\geqslant\left[\delta\left(\alpha_2\left(\alpha^{-1}\left(d\chi(||v||_\infty)+d\chi_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right)\right)\right)\right]\\
&\qquad\cdot P\left\{V(x(t\vee\tau_1),t\vee\tau_1)\geqslant\delta\left(\alpha_2\left(\alpha^{-1}\left(d\chi(||v||_\infty)+d\chi_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right)\right)\right)\right\}
\end{aligned}\tag{A12}
$$

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This together with (A11) gives

$$P\left\{V(x(t\vee\tau_1),t\vee\tau_1)\geqslant\delta\left(\alpha_2\left(\alpha^{-1}\left(d\chi(\|v\|_\infty)+d\chi_w\left(\sup_{t\geqslant 0}\|\Sigma(t)\Sigma(t)^{\mathrm{T}}\|\right)\right)\right)\right)\right\}$$
$$\leqslant\frac{2\alpha_2(\alpha^{-1}(d\chi(\|v\|_\infty)+d\chi_w(\sup_{t\geqslant 0}\|\Sigma(t)\Sigma(t)^{\mathrm{T}}\|)))}{\delta(\alpha_2(\alpha^{-1}(d\chi(\|v\|_\infty)+d\chi_w(\sup_{t\geqslant 0}\|\Sigma(t)\Sigma(t)^{\mathrm{T}}\|))))}\leqslant\epsilon'' \tag{A13}$$

where $\epsilon''\in(0,1)$ can be made arbitrarily small by an appropriate choice of $\delta\in\mathscr{K}_\infty$. Thus, by (4) and (A13), we have

$$P\left\{|x(t\vee\tau_1)|<\alpha_1^{-1}\left(\delta\left(\alpha_2\left(\alpha^{-1}\left(d\chi(\|v\|_\infty)+d\chi_w\left(\sup_{t\geqslant 0}\|\Sigma(t)\Sigma(t)^{\mathrm{T}}\|\right)\right)\right)\right)\right)\right\}\geqslant 1-\epsilon'' \tag{A14}$$

Let $\gamma(s)=\alpha_1^{-1}(\delta(\alpha_2(\alpha^{-1}(2d\chi(s)))))$ and $\gamma_w(s)=\alpha_1^{-1}(\delta(\alpha_2(\alpha^{-1}(2d\chi_w(s)))))$. Then, by simple calculations, it can be verified that for any $t\geqslant 0, x_0\in\mathscr{B}^c\backslash\{0\}$,

$$\begin{aligned}
&P\left\{|x(t)|<\beta(|x_0|,t)+\gamma(\|v\|_\infty)+\gamma_w\left(\sup_{t\geqslant 0}\|\Sigma(t)\Sigma(t)^{\mathrm{T}}\|\right)\right\}\\
&\geqslant P\left\{|x(t)|<\beta(|x_0|,t)+\alpha_1^{-1}\left(\delta\left(\alpha_2\left(\alpha^{-1}\left(d\chi(\|v\|_\infty)+d\chi_w\left(\sup_{t\geqslant 0}\|\Sigma(t)\Sigma(t)^{\mathrm{T}}\|\right)\right)\right)\right)\right)\right\}\\
&\geqslant P\Bigg\{\{|x(t\wedge\tau_1)|<\beta(|x_0|,t)\}\\
&\quad\cup\left\{|x(t\vee\tau_1)|<\alpha_1^{-1}\left(\delta\left(\alpha_2\left(\alpha^{-1}\left(d\chi(\|v\|_\infty)+d\chi_w\left(\sup_{t\geqslant 0}\|\Sigma(t)\Sigma(t)^{\mathrm{T}}\|\right)\right)\right)\right)\right)\right\}\Bigg\}
\end{aligned}$$

Combining this with (A5) and (A14) leads to

$$\begin{aligned}
&P\left\{|x(t)|<\beta(|x_0|,t)+\gamma(\|v\|_\infty)+\gamma_w\left(\sup_{t\geqslant 0}\|\Sigma(t)\Sigma(t)^{\mathrm{T}}\|\right)\right\}\\
&\geqslant\max\{1-\epsilon',1-\epsilon''\}\\
&=1-\min\{\epsilon',\epsilon''\}\triangleq 1-\epsilon\quad\forall t\geqslant 0,\ \ x_0\in\mathscr{B}^c\backslash\{0\}
\end{aligned} \tag{A15}$$

Case 2: $x_0\in\mathscr{B}\backslash\{0\}$. In this case $\tau_1=0$ a.s.

When $t>0$, $P\{t\in(\tau_1,\infty)\}=P\{t\in(0,\infty)\}=1$. Following the proof of *Case 1*, we know that (A14) still holds, and then,

$$\begin{aligned}
&P\left\{|x(t)|<\beta(|x_0|,t)+\gamma(\|v\|_\infty)+\gamma_w\left(\sup_{t\geqslant 0}\|\Sigma(t)\Sigma(t)^{\mathrm{T}}\|\right)\right\}\\
&=P\left\{|x(t)|<\beta(|x_0|,t)+\alpha_1^{-1}\left(\delta\left(\alpha_2\left(\alpha^{-1}\left(d\chi(\|v\|_\infty)+d\chi_w\left(\sup_{t\geqslant 0}\|\Sigma(t)\Sigma(t)^{\mathrm{T}}\|\right)\right)\right)\right)\right)\right\}
\end{aligned} \tag{A16}$$

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$$= P\left\{|x(t \vee \tau_1)| < \beta(|x_0|, t) + \alpha_1^{-1}\left(\delta\left(\alpha_2\left(\alpha^{-1}\left(d\chi(||v||_\infty) + d\chi_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right)\right)\right)\right)\right\}$$

$$\geqslant P\left\{|x(t \vee \tau_1)| < \alpha_1^{-1}\left(\delta\left(\alpha_2\left(\alpha^{-1}\left(d\chi(||v||_\infty) + d\chi_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right)\right)\right)\right)\right\}$$

$$\geqslant 1 - \epsilon''$$

When $t = 0$, by the definition of the set $\mathscr{B}$ and the definition of the function γ, we obtain

$$P\left\{|x(0)| < \beta(|x_0|, 0) + \gamma(||v||_\infty) + \gamma_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right\}$$
$$\geqslant P\left\{|x(0)| < \gamma(||v||_\infty) + \gamma_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right\} = 1$$

which implies

$$P\left\{|x(0)| < \beta(|x_0|, 0) + \gamma(||v||_\infty) + \gamma_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right\} = 1 \tag{A17}$$

Thus, by (A16) and (A17), we have

$$P\left\{|x(t)| < \beta(|x_0|, t) + \gamma(||v||_\infty) + \gamma_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right\} \geqslant 1 - \epsilon \quad \forall t \geqslant 0, \ \ x_0 \in \mathscr{B}\backslash\{0\} \tag{A18}$$

In conclusion, by (A15) and (A18), we have

$$P\left\{|x(t)| < \beta(|x_0|, t) + \gamma(||v||_\infty) + \gamma_w\left(\sup_{t\geqslant 0}||\Sigma(t)\Sigma(t)^{\mathrm{T}}||\right)\right\} \geqslant 1 - \epsilon \quad \forall t \geqslant 0, \ \ x_0 \in \mathbb{R}^n\backslash\{0\}$$

By causality, we obtain

$$P\left\{|x(t)| < \beta(|x_0|, t) + \gamma\left(\sup_{0\leqslant s\leqslant t}||v_s||\right) + \gamma_w\left(\sup_{0\leqslant s\leqslant t}||\Sigma(s)\Sigma(s)^{\mathrm{T}}||\right)\right\} \geqslant 1 - \epsilon \quad \forall t \geqslant 0, \ \ x_0 \in \mathbb{R}^n\backslash\{0\}$$

The proof is complete. □

APPENDIX B: PROOF OF LEMMA 1

As shown in Step 1 of Section 5, Lemma 1 holds with $k = 1$. Now, we demonstrate Lemma 1 by induction. Assume that Lemma 1 is true for Step $k - 1$, we will show that Lemma 1 is still true for Step k. For this purpose, consider the following function:

$$V_k = V_{k-1} + \frac{1}{2L^{2k-2}} z_k^2$$

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where

$$z_k = \hat{x}_k - \phi_{k-1}(\bar{\hat{x}}_{k-1}), \quad \phi_{k-1} = -Lb_{k-1}z_{k-1}$$
$$z_{k-1} = \hat{x}_{k-1} + Lb_{k-2}\hat{x}_{k-2} + L^2 b_{k-2}b_{k-3}\hat{x}_{k-3} + \cdots + L^{k-2}b_{k-2}b_{k-3}\cdots b_1\hat{x}_1$$

Then, it follows from (11) that

$$\begin{aligned}
dz_k &= \left[\hat{x}_{k+1} + L^k a_k \varepsilon_1 + Lb_{k-1}\left(\sum_{j=1}^{k-1}\frac{\partial z_{k-1}}{\partial \hat{x}_j}(\hat{x}_{j+1} + L^j a_j \varepsilon_1)\right)\right]\mathrm{d}t \\
&= \left[\hat{x}_{k+1} + L^k a_k \varepsilon_1 + \sum_{j=1}^{k-1} L^{k-j} b_{k-1}\cdots b_j (z_{j+1} - Lb_j z_j + L^j a_j \varepsilon_1)\right]\mathrm{d}t \\
&= [\hat{x}_{k+1} + L^k a_k \varepsilon_1 + L^{k-1} b_{k-1}\cdots b_1 (z_2 - Lb_1 z_1 + La_1\varepsilon_1) + L^{k-2} b_{k-1}\cdots b_2 (z_3 - Lb_2 z_2 + L^2 a_2 \varepsilon_1) \\
&\quad + \cdots + Lb_{k-1}(z_k - Lb_{k-1}z_{k-1} + L^{k-1}a_{k-1}\varepsilon_1)]\,\mathrm{d}t \\
&= (\hat{x}_{k+1} + L^k \tilde{d}_k \varepsilon_1 + L^k d_{k1} z_1 + L^{k-1} d_{k2} z_2 + \cdots + L d_{kk} z_k)\,\mathrm{d}t
\end{aligned}$$

where

$$\begin{aligned}
\tilde{d}_k &= a_k + b_{k-1}\cdots b_1 a_1 + b_{k-1}\cdots b_2 a_2 + \cdots + b_{k-1}a_{k-1} \\
d_{k1} &= -b_{k-1}\cdots b_1 b_1 \\
d_{k2} &= b_{k-1}\cdots b_2 b_1 - b_{k-1}\cdots b_2 b_2 \\
d_{kj} &= b_{k-1}\cdots b_j b_{j-1} - b_{k-1}\cdots b_j b_j, \quad j = 3, \ldots, k-1 \\
d_{kk} &= b_{k-1}
\end{aligned}$$

Thus, by Itô formula and Young inequality, we obtain

$$\begin{aligned}
\mathscr{L}V_k &= \mathscr{L}V_{k-1} + \frac{1}{L^{2k-2}} z_k(\hat{x}_{k+1} + L^k \tilde{d}_k \varepsilon_1 + L^k d_{k1} z_1 + L^{k-1} d_{k2} z_2 + \cdots + L d_{kk} z_k) \\
&\leqslant \mathscr{L}V_{k-1} + \frac{1}{L^{2k-2}} z_k(z_{k+1} + \phi_k) + z_k\left(\frac{1}{L^{k-2}}\tilde{d}_k \varepsilon_1 + \frac{1}{L^{k-2}} d_{k1} z_1\right. \\
&\quad \left. + \frac{1}{L^{k-1}} d_{k2} z_2 + \cdots + \frac{1}{L^{2k-3}} d_{kk} z_k\right) \\
&\leqslant \mathscr{L}V_{k-1} + \frac{1}{L^{2k-2}} z_k(z_{k+1} + \phi_k) + \left(\frac{\tilde{d}_k^2}{4L^{2k-3}} z_k^2 + L\varepsilon_1^2\right) + \left(\frac{d_{k1}^2}{4L^{2k-3}} z_k^2 + L z_1^2\right) \\
&\quad + \left(\frac{d_{k2}^2}{4L^{2k-3}} z_k^2 + \frac{z_2^2}{L}\right) + \cdots + \left(\frac{d_{k,k-1}^2}{4L^{2k-3}} z_k^2 + \frac{1}{L^{2k-5}} z_{k-1}^2\right) + \frac{1}{L^{2k-3}} d_{kk} z_k^2 \\
&= \mathscr{L}V_{k-1} + \frac{1}{L^{2k-2}} z_k(z_{k+1} + \phi_k) + z_k^2 \frac{1}{L^{2k-3}}\left[\frac{\tilde{d}_k^2}{4} + \frac{d_{k1}^2}{4} + \cdots + \frac{d_{k,k-1}^2}{4} + d_{kk}\right] \\
&\quad + L\varepsilon_1^2 + L z_1^2 + \frac{1}{L} z_2^2 + \cdots + \frac{1}{L^{2k-5}} z_{k-1}^2
\end{aligned} \tag{B1}$$

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This together with (22) leads to

$$\begin{aligned}\mathscr{L}V_k \leqslant & -[\delta(L-||P||^2)-2nC_e-kL]|\varepsilon|^2+2nC_e\left(\frac{|\hat{x}_{k+2}|^2}{L^{2k+2}}+\cdots+\frac{|\hat{x}_n|^2}{2L^{2n-2}}\right)\\ & +\Delta-(2nL-4nC_eb_1^2-kL)z_1^2-\sum_{j=2}^{k-1}\frac{1}{L^{2j-2}}[(n+j-k+1)L-4nC_eb_j^2]z_j^2\\ & +\frac{4nC_e}{L^{2k-2}}z_k^2+\frac{1}{4L^{2k-3}}z_k^2+\frac{1}{L^{2k-2}}z_k(z_{k+1}+\phi_k)\\ & +\frac{1}{L^{2k-3}}z_k^2\left[\frac{\tilde{d}_k^2}{4}+\frac{d_{k1}^2}{4}+\cdots+\frac{d_{k,k-1}^2}{4}+d_{kk}\right]+2nC_e\frac{|\hat{x}_{k+1}|^2}{L^{2k}}\\ & +\frac{1}{L}z_2^2+\cdots+\frac{1}{L^{2k-5}}z_{k-1}^2\end{aligned}\tag{B2}$$

Notice that

$$2nC_e\frac{|\hat{x}_{k+1}|^2}{L^{2k}}\leqslant 2nC_e\frac{|z_{k+1}+\phi_k|^2}{L^{2k}}\leqslant\frac{4nC_ez_{k+1}^2}{L^{2k}}+\frac{4nC_e\phi_k^2}{L^{2k}}$$

$$\frac{1}{L^{2k-2}}z_kz_{k+1}\leqslant\frac{L^{2k-3}z_{k+1}^2}{4L^{4k-4}}+\frac{1}{L^{2k-3}}z_k^2=\frac{z_{k+1}^2}{4L^{2k-1}}+\frac{1}{L^{2k-3}}z_k^2$$

Then, by (B2) we have[‡]

$$\begin{aligned}\mathscr{L}V_k \leqslant & -[\delta(L-||P||^2)-2nC_e-kL]|\varepsilon|^2+2nC_e\left(\frac{|\hat{x}_{k+2}|^2}{L^{2k+2}}+\cdots+\frac{|\hat{x}_n|^2}{2L^{2n-2}}\right)\\ & -(2nL-4nC_eb_1^2-kL)z_1^2-\sum_{j=2}^{k-1}\frac{1}{L^{2j-2}}[(n+j-k+1)L-4nC_eb_j^2]z_j^2\\ & +\frac{1}{L^{2k-2}}z_k\left[\phi_k+4nC_eLz_k+\frac{L}{4}z_k+\frac{L\tilde{d}_k^2}{4}z_k+\frac{Ld_{k1}^2}{4}z_k+\cdots\right.\end{aligned}\tag{B3}$$

[‡] When $k=n$, $z_{k+1}=0$ in (B1), we have

$$\begin{aligned}\mathscr{L}V_n \leqslant & -[\delta(L-||P||^2)-2nC_e-nL]|\varepsilon|^2-(nL-4nC_eb_1^2)z_1^2-\sum_{j=2}^{n-1}\frac{1}{L^{2j-2}}[(j+1)L-4nC_eb_j^2]z_j^2\\ & +\frac{1}{L^{2n-2}}z_n[\phi_n+L(b_n-n)z_n]+\frac{1}{L}z_2^2+\cdots+\frac{1}{L^{2n-5}}z_{n-1}^2+\Delta\end{aligned}$$

where $b_n=n+4nC_e+1/4+\tilde{d}_n^2/4+d_{n1}^2/4+\cdots+d_{n,n-1}^2/4+d_{nn}$.

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$$+\frac{Ld_{k,k-1}^2}{4}z_k + Ld_{kk}z_k + Lz_k\Bigg] + \frac{z_{k+1}^2}{4L^{2k-1}} + \frac{4nC_e z_{k+1}^2}{L^{2k}} + \frac{4nC_e\phi_k^2}{L^{2k}}$$

$$+\frac{1}{L}z_2^2 + \cdots + \frac{1}{L^{2k-5}}z_{k-1}^2 + \Delta, \quad k = 2, \ldots, n-1$$

Take the virtual control law

$$\phi_k = -Lb_k z_k, \quad k = 2, \ldots, n-1$$

where

$$b_k = n + 4nC_e + \frac{1}{4} + \frac{\tilde{d}_k^2}{4} + \frac{d_{k1}^2}{4} + \cdots + \frac{d_{k,k-1}^2}{4} + d_{kk} + 1, \quad k = 2, \ldots, n-1$$

Then, it follows from (B3) that

$$\begin{aligned}\mathscr{L}V_k \leqslant & -[\delta(L - ||P||^2) - 2nC_e - kL]|\varepsilon|^2 + 2nC_e\left(\frac{|\hat{x}_{k+2}|^2}{L^{2k+2}} + \cdots + \frac{|\hat{x}_n|^2}{2L^{2n-2}}\right)\\ & - (2nL - 4nC_e b_1^2 - (k+1)L)z_1^2 - \sum_{j=2}^{k}\frac{1}{L^{2j-2}}[(n+j-k)L - 4nC_e b_j^2]z_j^2\\ & + \frac{z_{k+1}^2}{4L^{2k-1}} + \frac{4nC_e}{L^{2k}}z_{k+1}^2 + \Delta\end{aligned}$$

The proof is complete. □

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